

ON CANTOR'S SINGULAR MOMENTS

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ABSTRACT. We evaluate a constant explicitly, thereby answering a question raised in [1].

In problem **10621** of the American Mathematical Monthly, Cantor's singular moments J_n were to be computed. In the published answers [1] they come out as

$$J_n = \frac{2}{3(n+1)} \sum_{j=0}^n \binom{n+1}{j} \frac{B_j}{3 \cdot 2^{j-1} - 1} \quad \text{for } n \geq 1$$

and $J_0 = 1$, with Bernoulli numbers B_n .

The editor asked, whether it is possible to compute

$$J_{-1} = \sum_{n \geq 0} J_n$$

exactly.

The purpose of this note is to do that. In [5] we considered a similar problem, and the gentle reader is invited to consult this paper for more background about the technique, as well as [6, 4] for more information about the Cantor distribution.

Following the method described in [3], we can write J_n as a contour integral viz.

$$\begin{aligned} J_n &= \frac{2}{3(n+1)} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(n+2)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1} ds \\ &= \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(n+1)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1} ds. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^N J_n &= J_0 + \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(N+2)\Gamma(1-s)}{\Gamma(N+2-s)s} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1} ds \\ &\quad - \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(2)\Gamma(1-s)}{\Gamma(2-s)s} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1} ds. \end{aligned}$$

From this form, one could even compute the asymptotics as $N \rightarrow \infty$. However, here, we only have to note that the first integral is of order $N^{1-\log_2 3}$, which means that it

goes to zero. Consequently

$$\begin{aligned}
\sum_{n \geq 0} J_n &= 1 - \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{(1-s)s} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1} ds \\
&= \frac{4}{3} + \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} \frac{\zeta(s)}{3 \cdot 2^{-s} - 1} ds \\
&= \frac{4}{3} + \frac{2}{3} \sum_{k,m \geq 1} 3^{-k} \cdot \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} \left(\frac{2^k}{m}\right)^s ds.
\end{aligned}$$

The last step was by using the Dirichlet series for $\zeta(s)$ and the geometric series, both valid for $\Re s = \frac{3}{2}$. A simple application of residue calculus, as it is often used in the context of the Mellin–Perron summation formula (see [7, 2]) evaluates the integrals inside the summation:

$$\frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} t^s ds = \begin{cases} t-1 & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases}.$$

Therefore

$$\begin{aligned}
\sum_{n \geq 0} J_n &= 1 + \frac{2}{3} \sum_{k \geq 1} \sum_{1 \leq m \leq 2^k} 3^{-k} \left(\frac{2^k}{m} - 1\right) \\
&= 1 + \frac{2}{3} \sum_{k \geq 1} \left(\frac{2}{3}\right)^k H_{2^k} - \frac{2}{3} \sum_{k \geq 1} \left(\frac{2}{3}\right)^k \\
&= -\frac{1}{3} + \frac{2}{3} \sum_{k \geq 1} \left(\frac{2}{3}\right)^k H_{2^k} = 3.36465\ 07281\ 00925\ 16083\ 89349\ 6289 \dots,
\end{aligned}$$

with harmonic numbers $H_n = \sum_{1 \leq k \leq n} \frac{1}{k}$.

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